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A new approach to measuring discretization error

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ABSTRACT: A fuzzy-measure procedure is proposed to introduce *a priori* error indicator for estimating the errors of finite element solutions due to the mesh discretization. This indicator provides an effective global estimate of the relative error norm and the procedure is very simple for practical engineering applications.

1 INTRODUCTION

The types of error that may occur in finite element solutions are due to

1. Rounding-off,
2. Uncertainty of material parameters, and
3. Mesh discretization.

In this paper we will be concerned with calculating the discretization error based on the fuzzy-measure approach. It is important to estimate such errors in order to improve the finite element solutions to engineering problems. There are two methods for calculating this type of error: *a priori* error estimate which predicts the solution error before a particular finite element analysis is carried out, and *a posteriori* estimate which measures the error after the finite element solution is obtained.

The theory of fuzzy measures introduced by Sugeno (1977) is used to propose *a priori* error indicator which can provide a global measure of the relative error norm in the mesh discretization of a continuum. Since the term "fuzzy" may convey a negative connotation, a word of caution is desired to be made here before the mathematical properties of fuzzy measure are presented in the subsequent section. The concept of fuzzy measure is based on that of the fuzzy-set theory (Zadeh, 1965) which deals with sets having fuzzy boundary and makes mathematics work in the sphere of imprecision. Fuzzy measure is a set function used to calculate the uncertainty in the transition of fuzzy phenomena. There are no implications that the mathematical analysis of fuzzy measure is fuzzy and there are no intentions that fuzzy measures are used to make precise systems imprecise.

2 FUZZY MEASURE

Let X be a non-empty set and $\mathcal{B}$ be a Borel field of X. A fuzzy measure g which is a set function defined on $\mathcal{B}$ has the following properties.

P1. Boundary conditions: $g(\emptyset) = 0$, $g(X) = 1$.

P2. Monotonicity: If $A, B \in \mathcal{B}$ and $A \subset B$, then $g(A) \leq g(B)$.

P3. Continuity: If $F_n \in \mathcal{B}$ for $1 \leq n < \infty$ and a sequence $\{F_n\}$ is monotone, then $\lim g_{n\to\infty}(F_n) = g(\lim_{n\to\infty} F_n)$.

Based on the property P2, the g_λ-fuzzy measure which satisfies the additional condition: $A, B \in \mathcal{B}$ and $A \cap B = \emptyset$ is introduced as

$$g(A \cup B) = g(A) + g(B) + \lambda\, g(A)\, g(B) \qquad (1)$$

in which $\lambda \in (-1, +\infty)$

It is noted that ordinary measures in the theory of Lebesgue integrals have additivity, but fuzzy measure as expressed in Eq.(1) is additive only when $\lambda = 0$. Hence, fuzzy measure is a generalization of ordinary measure.

Let $X = \{x_1, x_2, ..., x_n\}$ be a finite set and g_λ be a fuzzy measure of X. The fuzzy density function of g_λ is defined as a mapping of x_i into $g(x_i)$ such that

$$g_\lambda(\{x_i\}) = g^i, \qquad i = 1, n. \qquad (2)$$

where g^i is a simplified notation of $g(x_i)$.

In general, the fuzzy measure of the finite set X can be defined by

$$g_\lambda(X) = \sum_{i=1}^{n} g^i + \lambda \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} g^i g^j + ... + \lambda^{n-1} g^1 g^2 ... g^n \qquad (3)$$

When $\lambda \neq 0$, ie. $\Sigma^n_{i=1}\, g^i \neq 1$, Eq.(3) can be rewritten as

$$g_\lambda(X) = \frac{1}{\lambda}\left[\prod_{i=1}^{n}(1 + \lambda g^i) - 1\right] \qquad (4)$$

With $g(X) = 1$ (property P1), the value of λ can be obtained from the following equation.

$$\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda g^i) \tag{5}$$

The bounds of λ depend on the sum of the fuzzy densities such as
$\lambda \,\varepsilon\, [-1, 0]$ for $\Sigma_{i=1}^{n} g^i > 1$ and
$\lambda \,\varepsilon\, [0, +\infty]$ for $\Sigma_{i=1}^{n} g^i < 0$.
In this present context, the meaning of the fuzzy densities is interpreted as the grade of importance of x_i towards the total evaluation. Further studies on fuzzy measures can be found in (Banon, 1981) and (Wierzchon, 1982).

3 DISCRETIZATION ERROR AND MEASURES OF ERROR

3.1 Errors in displacements and stresses

Consider a linear elastic problem which may be expressed in the form (Zienkiewicz & Zhu, 1987):

$$[L]\{u\} - \{q\} = 0 \tag{6}$$

in a domain Ω with boundary Γ, and under prescribed displacement condition

$$\{u\} = \{\bar{u}\} \tag{7a}$$
on the boundary Γ_u

and prescribed traction

$$[M]\{u\} = \{\bar{t}\} \tag{7b}$$
on the boundary Γ_t
where $\Gamma_u \cup \Gamma_t = \Gamma$, and $\Gamma_u \cap \Gamma_t = \emptyset$

[L] and [M] in Eqs. (6) and (7b) are linear operators such that

$[L] = [S]^T[D][S]$ and
$[M] = [G][D][S]$
in which [S] is the matrix operator defining the relation of the displacements {u} and strains {ε} as

$$\{\varepsilon\} = [S]\{u\} \tag{8}$$

and [D] is the elasticity matrix which defines the stress vector {σ} as

$$\{\sigma\} = [D]\{\varepsilon\} \tag{9}$$

and [G] is a linear operator which transfers stresses into forces along the boundary of Ω.
Using the finite element formulation, an approximation to the solution of the above problem for {u} can be given by

$$\{u\} \approx \{\hat{u}\} = [N]\{\tilde{u}\} \tag{10}$$
where [N] is the interpolation function and $\{\tilde{u}\}$ is the nodal displacement vector. $\{\tilde{u}\}$ can be determined by

$$[K]\{\tilde{u}\} = \{f\} \tag{11}$$
where

$$[K] = \int_{\Omega} [B]^T\,[D]\,[B]\,d\Omega$$

in which [B] = [S][N]
and

$$\{f\} = \int_{\Omega} [N]^T\{q\}\,d\,\Omega + \int_{\Gamma_t} [N]^T\,\{\bar{t}\}\,d\,\Gamma$$

The stresses obtained from the finite element solution are:

$$\{\sigma\} \approx \{\hat{\sigma}\} = ([D][B])\,\{\tilde{u}\} \tag{12}$$

The discretization error is defined as the discrepancy between the approximate (finite element) and the exact solutions.
Therefore, the error for displacement at a point will be

$$e_u = u - \hat{u} \tag{13a}$$

and for a stress value

$$e_\sigma = \sigma - \hat{\sigma} \tag{13b}$$

However, the pointwise definition of errors as given in Eqs. (13a) & (13b) is difficult to specify. Thus, other error estimates are needed to overcome this difficulty.

3.2 An error indicator

A discretization error can be expressed in the form (Cook *et. al.*, 1989)

$e = O(h^{q-r})$
where
O stands for "order",
h is the "characteristic length" of an element,
q is one plus the degree of the highest complete polynomial in the element displacement field.
r = 0 for displacement error and r = 1 for stress error, in elasticity.
As a result, an error measure can be expressed as follows.

$$e \approx Ch^{q-r} \tag{14}$$
where C is a problem-dependent constant and the other terms are defined previously.

On an averaging process, a general error estimate can be obtained in terms of the norm (such as the L_2 norm)

$$e = \|\Phi - \Phi^*\| \tag{15}$$

where Φ and Φ^* are the exact and approximated values respectively.

With the values of Φ and Φ^* at the points p_1, p_2, ..., p_n, the corresponding discrete version of the L_2 norm of the above error measure will be

$$e=\|\Phi-\Phi^*\|=\left(\sum_{i=1}^{n} |\Phi(p_i)-\Phi^*(p_i)|^2\right)^{1/2} \tag{16}$$

And the relative error norm will be

$$\|e_r\| = \frac{\|\Phi - \Phi^*\|}{\|\Phi\|} \tag{17}$$

A corresponding global estimate of relative error was suggested as (Cook *et. al.*, 1989)

$$\|e_r\| = \rho_1\rho_2\, \ell^{\,q-r} \tag{18}$$

in which ρ_1 is the largest element aspect ratio which is the ratio of the largest length to the smallest length of an element, ρ_2 is the ratio of the characteristic length of the largest element to that of the smallest element, and ℓ is the dimensionless length defined by

$\ell = 1/(K^{1/n})$

where K is the number of elements in the mesh, and n is the spatial dimension such as n = 1 for line, 2 for plane, and 3 for solid problems.

Besides the simplicity of the error indicator as given in Eq.(18), it gives rough prediction and in some cases it is unreliable.

4 THE PROPOSED ERROR INDICATOR

4.1 The concept

As the theory of fuzzy measures provides a procedure for measuring the transition of fuzziness between the relation of ordinary sets, the value of λ in Eq.(5) will be treated as an index for measuring the efficiency of the union of disjoint subsets (whose intersection is an empty set). It is noted that the error indicator as given in Eq.(18) usually underestimates the magnitude of error when the mesh is uniformly discretized, but may overestimate the error when the ratios ρ_1 and ρ_2 becomes larger than one. In order to avoid this inconsistency, the modeling is suggested with the following steps.

1. The dimensionless length ℓ is treated as a governing parameter. ρ_1 is a redundant parameter. The ratio ρ_2 (from now ρ_2 will be denoted as ρ) is taken into account implicitly, ie. in the fuzzy-measure procedure.
2. Applying the procedure of g_λ-fuzzy measures to calculate the fuzzy transition between ρ, ℓ and (q-r) on an averaging process.
3. Finally, the average measure of the fuzzy transition will be added to ℓ to provide the estimate of error.

4.2 The formulation

An averaging process for measuring the fuzzy transition between the parameters is defined using the L_2 norm

$$\mu = \left(\sum_{i=1}^{n} |g(A_i) - w(A_i)|^2\right)^{1/2} \tag{19}$$

where

μ is called the average measure of fuzzy transition,

$A_i = \{p_1, p_2, ..., p_i\}$,

$g(A_i)$ can be obtained from Eq.(4) after solving for λ, and

$w(A_i) = w_1 + ... + w_i$.

Thus, the new estimator of the relative discretization error will be defined by adding μ to the governing parameter ℓ such that

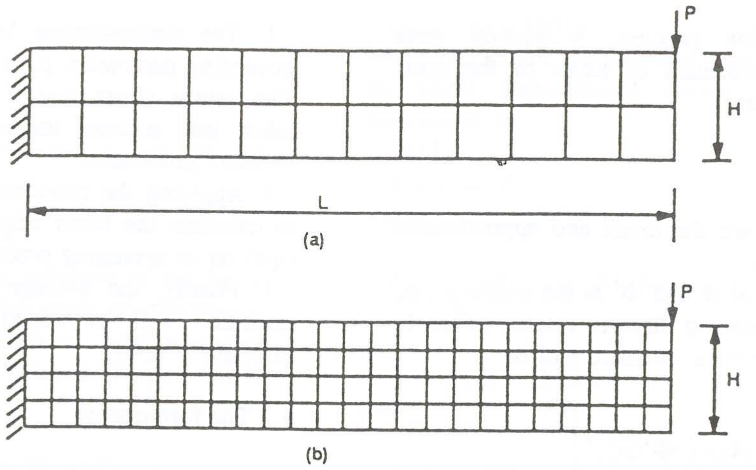
$$\|e^*_r\| = (\ell+\mu)^{\beta} \tag{20}$$

in which $\|e^*_r\|$ is the new error norm estimate, ℓ is defined previously and $\beta = (q-r)$.

To simplify the procedure for calculating μ in Eq.(20), only three fuzzy densities will be introduced. Therefore, Eq.(19) will lead to

$$\mu = \left[(\lambda g^1 g^2)^2 + (\lambda g^2 g^3)^2 + (\lambda g^3 g^1)^2\right]^{1/2} \tag{21}$$

in which g^1 is the fuzzy density of ρ, g^2 is the fuzzy density of ℓ, and g^3 is the fuzzy density of (q-r). These fuzzy densities are also interpreted as *a priori* weights of the corresponding parameters in Eq.(18). The equations for calculating these weights can be defined as



Figs. 1(a)&(b) - Finite element meshes of a cantilevered beam

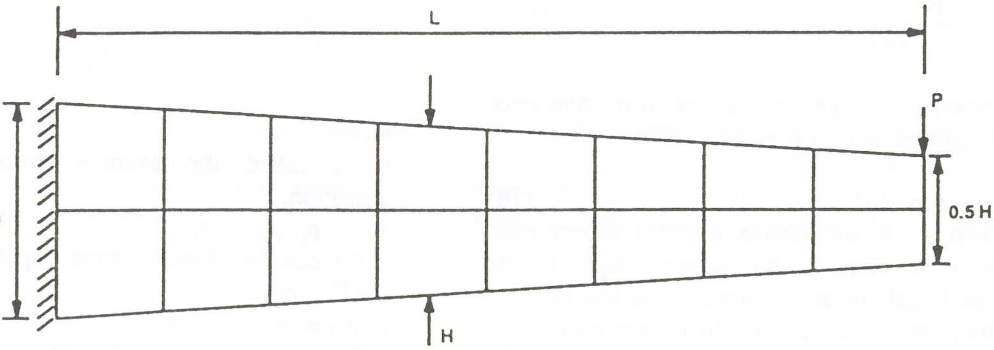


Fig. 2 - Finite element mesh of a tapered cantilevered beam

$$w_1 = 0.25 - \left[\frac{0.25(\rho - 1)}{\rho}\right] \qquad (22)$$

for $w_1 \, \varepsilon \, [0, 0.25]$

w_2 is set up following the criterion for ℓ

$$w_2 = 0.5 - \left(\frac{0.5}{K^{1/n}}\right) \qquad (23)$$

for $w_2 \, \varepsilon \, [0, 0.5]$

and

$$w_3 = 0.25 - (0.25)^q \qquad (24)$$

for $w_3 \, \varepsilon \, [0, 0.25]$

The above equations imply that $w_1 = g^1$, $w_2 = g^2$ and $w_3 = g^3$.

And ρ is calculated on an averaging process rather than an extreme ratio, ie.

$$\rho = \frac{\sum_{i=1}^{K} \rho_i}{K} \qquad (25)$$

where ρ_i is the aspect ratio of each element i and K is again the number of finite elements in the mesh.

In the cases of quadratic elements, it is observed that the measures of errors will improve if q = 2.2 instead of being 3 as suggested in Eq.(16). The former value will be used in the following examples for the error estimates.

5 NUMERICAL EXAMPLES

Figure 1(a) shows a cantilevered beam carrying a vertical load at its free end. The length (L) of the beam is 1.2m, the width b is 0.2m, and the depth (H) is 0.2m. The elastic modulus (E) of the beam is 8 GPa, the Poisson ratio is assumed to be zero and the end load (P) is 10 kN. The beam is discretized into 24 quadrilateral elements of equal length h = 0.1m. Plane-stress condition is applied.

The fuzzy densities calculated from Eqs.(22-24) are: $g^1 = 0.25$, $g^2 = 0.398$ and $g^3 = 0.188$. Using Eq.(5) and Eq.(21) we have $\lambda = 0.702$ and $\mu = 0.093$ respectively. Then the measures of discretization error for two different types of element are obtained as follows.

- *For bilinear elements*:

The global estimate of the relative error norm for vertical displacements is obtained from Eq.(20) as

$\|e^*_r\| = 8.8\%$.

While from Eq.(18), $\|e_r\| = 4.2\%$ and from the beam theory, $\|e'_r\| = 9.7\%$ ($\|e'_r\|$ stands for a relative error norm where the exact solution is based on the beam theory)

For maximum bending stresses:

$\|e^*_r\| = 29.7\%$

While

$\|e_r\| = 20.4\%$ and $\|e'_r\| = 30.6\%$

- *For quadratic elements*:

The error for vertical displacements is estimated as

$\|e^*_r\| = 5.7\%$

$\|e_r\| = 0.85\%$, and $\|e'_r\| = 6.9\%$.

For maximum bending stresses:

$\|e^*_r\| = 20.9\%$

$\|e_r\| = 4.2\%$, and $\|e'_r\| = 21.4\%$.

For another example, Figure 1(b) shows a beam which carries the same load and has the same geometrical and material properties of that in Figure 1(a) except that the mesh is uniformly divided into 96 bilinear elements. The results are:

$g^1 = 0.250$, $g^2 = 0.449$, $g^3 = 0.234$

$\lambda = 0.237$, $\mu = 0.039$

Then the relative error norm for displacements is

$\|e^*_r\| = 1.99\%$

$\|e_r\| = 1.04\%$, and $\|e'_r\| = 1.28\%$.

And for stresses:

$\|e^*_r\| = 14.10$

$\|e_r\| = 10.20\%$, and $\|e'_r\| = 13.35\%$.

As a third example, Figure 2 shows a tapered cantilevered beam. The beam is 1.6m long and 0.1m wide. The initial depth H_0 is 0.4m and the depth at the end is 0.2m. The elastic modulus (E) of the beam is 200 GPa, and the end load P = 10 kN. Again the Poisson ratio is assumed to be zero. The results are obtained as

$g^1 = 0.172$, $g^2 = 0.375$, $g^3 = 0.187$

$\lambda = 1.441$, $\mu = 0.145$

For vertical displacements:

$\|e^*_r\| = 15.60$ %

$\|e_r\| = 25.00\%$, and $\|e'_r\| = 12.97\%$.

And for stresses:

$\|e^*_r\| = 39.50\%$

$\|e_r\| = 100\%$, and $\|e'_r\| = 35.08\%$.

6. CONCLUSION

A fuzzy-measure approach has been used to propose a global estimate of relative discretization error in an effective and systematic way. The procedure presented in this paper is simple for practical engineering applications. This error indicator is observed to be of high quality, particularly when the mesh is regularly divided. However, improvement on the calculations of the fuzzy densities may be further investigated in future study. And the correction of the parameter q for the case of quadratic elements may need more rigorous formulation as it is only based on heuristic reasoning at present.

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