GROUNDWATER DRAWDOWN RELATED SETTLEMENT AND PUMP TEST INTERPRETATION

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ABSTRACT

Settlement associated with groundwater drawdown can be an important design factor for construction below the watertable. As a result, prediction of the extent and magnitude of groundwater drawdown is an essential part of the evaluation of potential impacts of deep structures. Pump test analysis is routinely employed to for assessment of hydraulic properties of aquifer units affected by deep excavation. Interpretation of pump tests makes use of the rate of groundwater extraction and the drawdown measured in the pumping bores and at other locations. Settlement is not usually considered during interpretation of pump tests. This paper discusses settlement due to dewatering and proposes use of settlement monitoring for interpretation of pump test results.

1 INTRODUCTION

Prediction of groundwater impacts arising from deep excavations is an essential part of the excavation design for excavations below the groundwater table. It involves assessment of groundwater inflow and prediction of drawdown response and generally is a requirement for development approvals. In mining applications, the same basic considerations apply.

Assessments involve development of a conceptual model drawn from the geological interpretation of ground conditions and monitoring of pre-development groundwater levels to evaluate the depth of drawdown and background variation in groundwater levels in response to factors such as rainfall, tidal effects and nearby activities. Where existing developments are present below the groundwater there may be variations in groundwater level associated with basement drainage in rural settings there may be drawdown associated with groundwater use for irrigation. In this paper, settlement due to drawdown is discussed and the use of settlement monitoring to improve the interpretation of test pumping is described.

2 SETTLEMENT DUE TO DEWATERING

Settlement is often a factor of primary important in assessment of the groundwater impacts associated deep excavations where compressible sediments are present. In Melbourne, Australia, settlement associated with construction of major building developments and civil infrastructure has resulted in observed settlement and lead to protracted legal disputes. Groundwater seepage to the Burnley Tunnel in South Melbourne results in continuing significant seepage and permanent groundwater injection is used to prevent ongoing settlement of the compressible Coode Island Silt. In Sydney, settlement associated with dewatering during construction of the Eastern Distributor resulted in damage distance from the works areas and construction of the Northside Storage Tunnel resulted in settlement beneath Middle Harbour, which raised concerns regarding potential impacts on a main sewer pipeline along the harbour floor supported on piles into the channel sediments.

There are many other examples where deep excavation or groundwater extraction as resulted in substantial settlement. Well-known examples are subsidence in Venice and Bangkok. The following examples are among a large number noted by Poland (1984):

- Latrobe Valley, Victoria. Settlement of in excess of 1.6 m has been recorded around coalmines in a thick shallow brown coal deposit with settlement in excess of 200 mm over an area of 100 km^2 as a result of groundwater extraction to prevent floor heave.
- Shanghai, China. Settlement of 2.6 m occurred from 1921 to 1965 within the upper 300 m of freshwater and marine sediments due to extraction of water for domestic and industrial use.
- Taipei Basin, Taiwan. Settlement of 1.9 occurred between 1955 and 1974 because of groundwater extraction from a deep alluvial deposit.

One example of groundwater drawdown related settlement of major settlement due to groundwater extraction in California occurred in Antelope Valley 80 km northeast of Los Angeles. Hoffmann *et al* (2003) describe regional land subsidence within a basin filled to depths of more than 2 km with fluvial and lacustrine sediments used to supply water for agriculture and town water. Settlement in the aquifer is estimated to be up to 2.5 m since the start of groundwater extraction. Groundwater use commenced following an investigation in 1911, which indicated water resources, which could support substantial agricultural development. As groundwater levels fell, agricultural use became uneconomic and currently use is primarily for municipal and industrial purposes. Figure 1 from Hoffmann *et al* (2003) illustrates the drawdown and settlement, which has occurred.

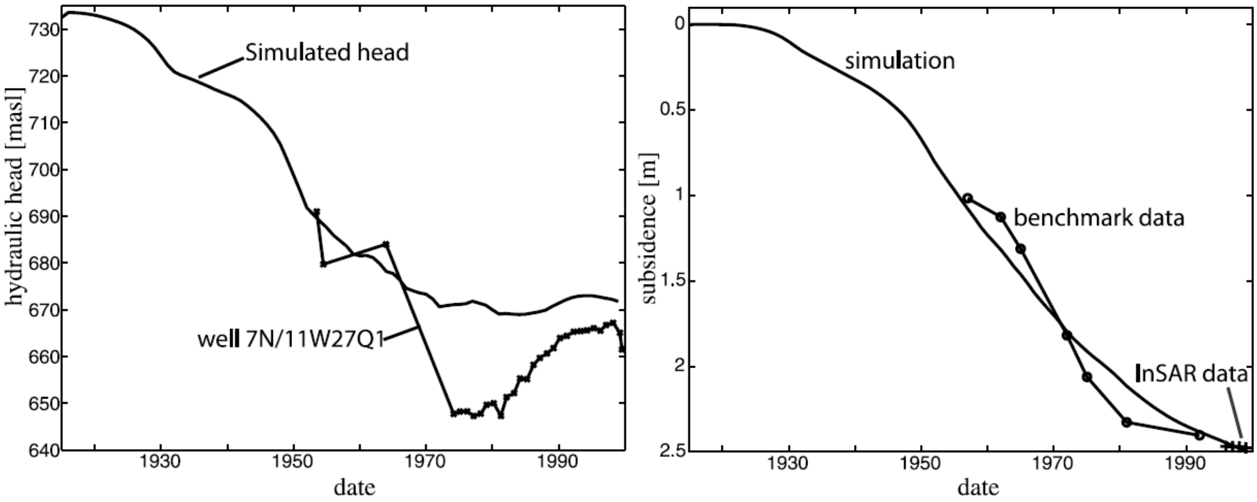

Figure 1: Groundwater drawdown and subsidence in Antelope Valley (Hoffmann *et al* 2003)

Hoffmann *et al* carried out a study using ground movement records from extensometer monitoring and InSAR satellite monitoring combined with groundwater monitoring to assess storage characteristics of the ground profile and in particular inelastic components of storage. Three high yielding aquifers are present separated by compressible low yielding sediments. Water is drawn from the upper unconfined aquifer and the middle aquifer. The study made use of conventional groundwater modelling to assess distribution of groundwater level changes in the main aquifer and one-dimensional consolidation theory to interpret delays in response. The study demonstrated the utility of InSAR monitoring to measure ground movement over wide areas. The InSAR measurements were in good agreement with the extensometer results. The results illustrated in Figure 2 demonstrate the link between annual groundwater level variations and ground movement. The InSAR measurements show good resolution of the pattern of movement over a 20 mm range over a period of four years and indicate movement below the level of anchorage of the reference extensometer (anchored at 363 m depth). Based on the results of the study zones of rapid and slow response to groundwater level changes were identified ranging from essentially instantaneous to over 200 years and estimates of inelastic storage coefficients were made.

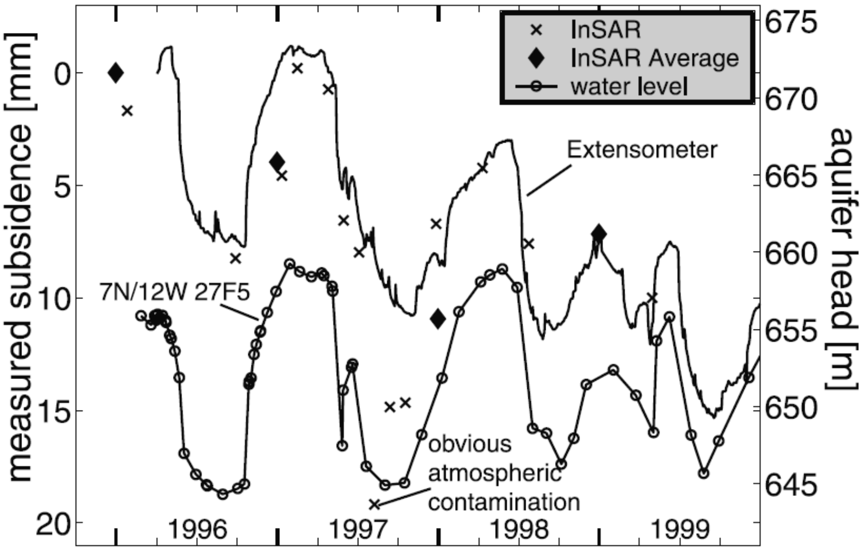

Figure 2: Monitoring of drawdown and subsidence - Antelope Valley, California (Hoffmann *et al* (2003)

During construction of the Sydney Northside Storage Tunnel to carry sewer overflows, seepage to the part of the tunnel in sandstone rock beneath a palaeochannel at Middle Harbour resulted in significant drawdown beneath the 70 m thick palaeochannel sediments and the development of settlement over a period of five years.

During investigation of the area, test pumping was carried out to provide information to predict behaviour during tunnelling the test involve pumping for a well screened in rock and monitoring of piezometric response in the rock and in the base of palaeochannel. In addition, settlement was monitored at a series of locations. Settlement of 5 mm occurred during the test pumping. This result, together with the piezometric response, allowed prediction of settlement for nominated inflow rates and was an important input to the design of a grouting program to reduce the rate of seepage to the tunnel.

The settlement resulted in movement of a sewer main installed across the floor of Middle Harbour supported by friction piles into the channel sediments. Figure 3 (after Best and Parker 2005) shows the variation in drawdown at the base of the palaeochannel and the measured settlement over a five-year period spanning construction and operation of the Northside Storage Tunnel. A major programme of grouting was carried out during tunnelling of a section at the floor of palaeochannel where the sandstone was broken as a result of valley bulging. The effects of grouting after tunnel construction are apparent in the record resulting in a sharp reduction in drawdown. The drawdown reduced briefly during periods of surcharge when the tunnel was filled with sewer overflows during high rainfall events while the treatment plant at North Head treated the accumulated flows. It is also interesting to note the gradual reduction in drawdown over time after construction indicating seepage to the tunnel gradually reduced as a result of a clogging process. Settlement stabilised at about 85 mm for the result illustrated in Figure 3 though higher settlements were recorded (up to about 110 mm) at other locations.

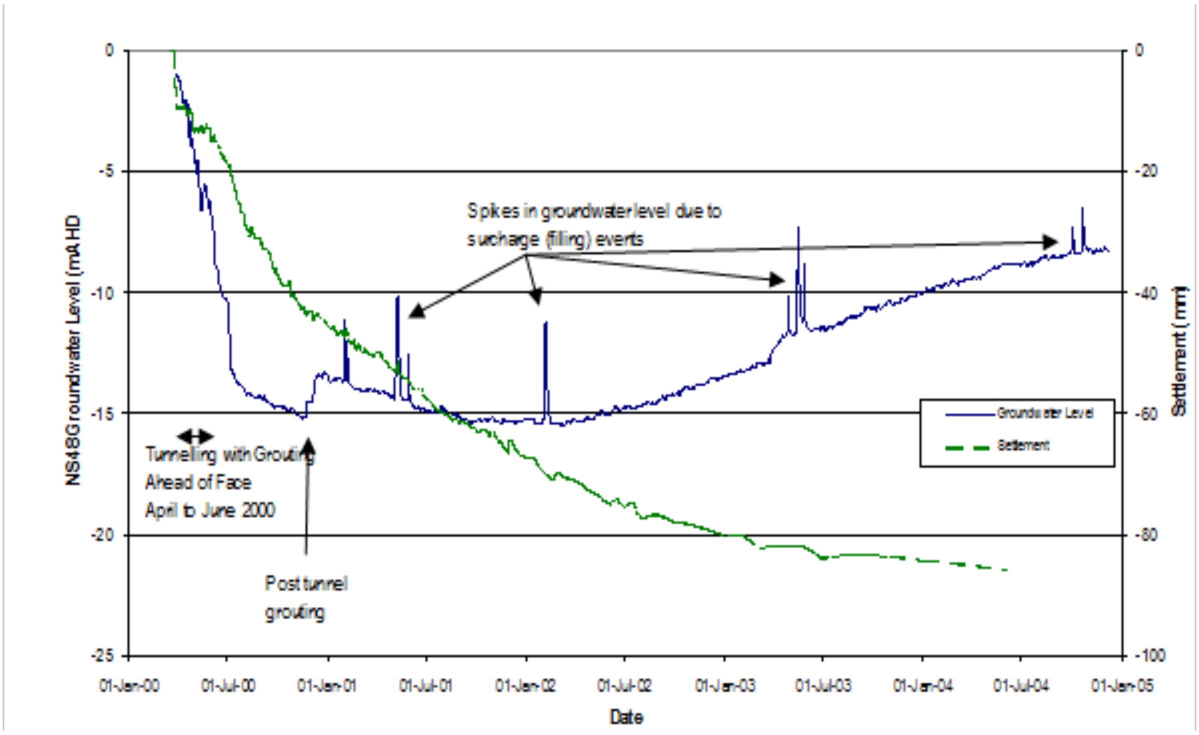

Figure 3: Drawdown and settlement - Middle Harbour Palaeochannel due to Northside Storage Tunnel

Coal seam gas extraction is being carried out in the Surat and Bowen Basins in Queensland. This involved extraction of groundwater to reduce pore pressure at depths of up to several hundred metres to release the gas (methane and propane) contained in coal seams. This groundwater extraction results in settlement and interferometric synthetic aperture radar (InSAR) methods are used to track settlement over vast areas. The power of this method, which involves comparison of phase difference in satellite radar images of the ground surface is illustrated in a report by Altamira (2012) which provided a baseline assessment of background settlement. Figure 4 shows the extent of the coverage where settlement monitoring over a four year period was interpreted from satellite images.

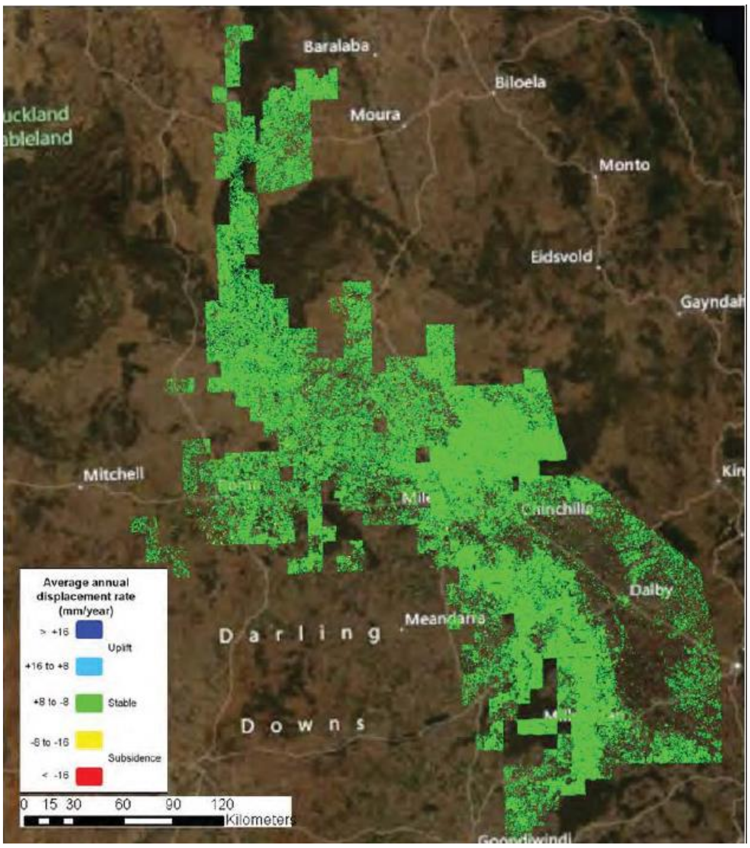

Figure 4: Coverage of InSAR settlement monitoring - Surat Basin (Altamira 2012)

The coverage of the method is vast and movements with sub-10 mm resolution are discriminated. Figure 5 taken from the Altamira report illustrates the response interpreted at one location selected from the area covered in Figure 4. Each settlement point represents a pass of the monitoring satellite.

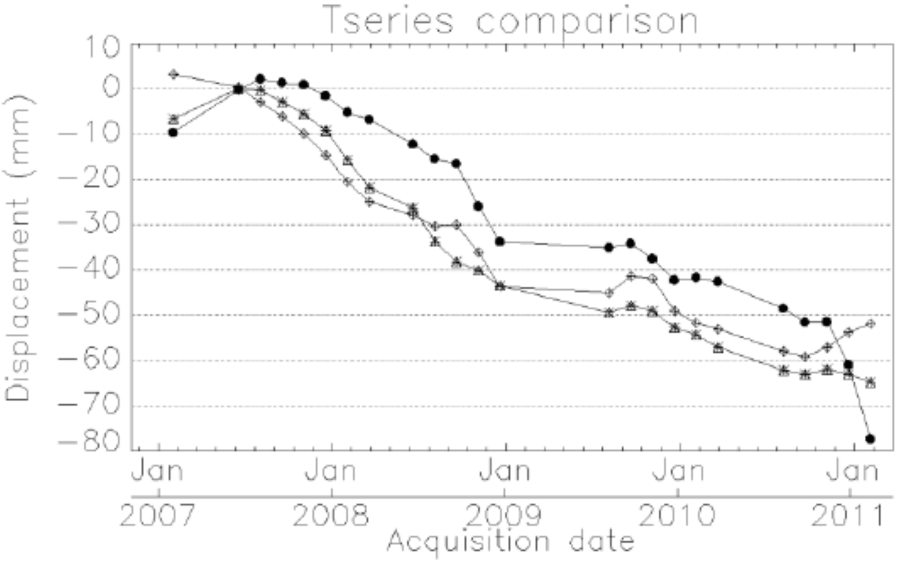

Figure 5: Selected interpretation record using InSAR (Altamira 2012)

Averaging methods are used to reduce scatter and some areas are unsuitable for interpretation such as areas of ploughed fields. The capacity to resolve movement over such a wide area provides a powerful tool, which is being used to interpret movement associated with coal seam gas extraction.

3 APPROACH FOR EXTENDING PUMP TEST ANALYSIS

Groundwater investigations are typically in the form of testing of response to changes in conditions at monitoring bores. This can be in the form of packer testing, falling or rising head testing or response to test pumping. Standard methods are employed for interpretation of these tests. In the case of interpretation of response to test pumping a wide range of interpretation methods are available depending upon the hydrogeological model of the groundwater system. Kruseman and De Ridder (2000) provide an excellent description of method for interpretation of test pumping results, which can be applied to conditions ranging from extensive confined aquifers, to leaky aquifers and aquifers bounded by low or high permeability features. Interpretation software is available which can carry our automatic curve fitting using a variety of hydrogeological models.

Pump testing is often carried out to assess hydraulic parameters for design for major mining and civil infrastructure developments. It provides a means of assessing the transmissivity of dominant aquifers and can provide some information about geological units overlying or underlying the dominant aquifer.

3.1 USE OF SETTLEMENT MONITORING TO ENHANCE TEST INTERPRETATION

Use of settlement monitoring to extend interpretation of the results for test pumping is proposed. This would provide more complete use of spatial response induced by testing. Typically, test pumping involves a pumping bore and a small number of monitoring piezometers. The results of testing provide limited information regarding spatial response. In formations which are extensive and hydraulic properties are expected to be consistent testing for a pumping bore with a few monitoring bores would typically provide a sound basis for interpretation of the properties of a single aquifer and some information regarding overlying or underlying units of lower permeability. Where there is strong directional behaviour as can occur in palaeochannels or in faulted rock systems, multiple monitoring piezometers are required to provide information regarding the distribution of the drawdown. In urban settings, the locations available for installation of monitoring bores are typically limited and delays in securing approval for installation can be problematic.

The paper proposes use of settlement monitoring to extend the interpretation of test pumping results. This would involve monitoring of settlement at high resolution in the area surrounding the test. This offers a number of potential benefits:

- Identification of directional behaviour (such as that resulting from the present of structures such as dykes and faults)
- Interpretation of vertical consolidation characteristics of compressible units overlying the aquifer tested
- Improvement of the overall interpretation by allowing clearer discrimination of the radial extent of response
- Allowing improvement in test analysis where there are constraints on installation of monitoring piezometers.

Surface settlement monitoring only requires access to the ground surface and typically involves minimal interference to other activities. Approvals for conduct of conventional survey are much less restrictive than those required for installation of monitoring piezometers.

In order to take advantage of settlement monitoring to extend the results of test pumping, analytical tools beyond those normally employed for test pump interpretation are required. Theory is available which allows prediction of drawdown for extensive layered aquifer systems subject to pumping. These are readily extended to allow prediction of surface settlement. Approaches by others have included employment of standard consolidation theory to assess surface settlement from dewatering. An approach described by Niu *et al* (2013) involved assessment of settlement associated with vertical consolidate of a compressible unit overlying an aquifer in which drawdown was considered stable with time in the aquifer and settlement of the overlying compressible sediment was obtained by integration. The results were for use in estimation of surface settlement at the pumping bore.

An analysis method for calculation of drawdown in a layered aquifer system was developed by Neuman (1975) and a similar approach was followed by Booker and Best (1990) and Best and Booker (2000) who extended the application of this method for calculation of settlement associated with groundwater extraction. These methods allow calculation of drawdown in a system of aquifers separated by lower permeability beds. The method involves solution of the problem in the Laplace transform domain and numerical inversion to obtain the result in the time domain.

There are many methods for numerical Laplace inversion. Two which have been demonstrated to be effective for layered groundwater flow are Talbot's method (1939) and the Gaver-Stehfast method (Gaver 1966). This process involves evaluation of the result for selected values of the transform variable and combining these results using inversion algorithms to obtain results at particular times. The Talbot method involves evaluation for a number of complex values of the transform variable while the Gaver-Stehfast has the advantage that evaluations are for real values of the transform variable.

Layered groundwater flow models predict drawdown and flow from the low permeability beds to the aquifers and settlement is obtained analytically using the integral of flow from the low permeability beds and the head reduction in the aquifers. For soil strength material, it is reasonable to approximate settlement as the volume of water removed from a soil column below the watertable. The theory is provided in Appendix A setting out the formulation. For rocks a minor adjustment is required to account for compressibility of water and Biot's coefficient linking settlement with pore pressure change.

An example is illustrated in Figure 6 for the case of a transmissive aquifer overlain by a compressible unit.

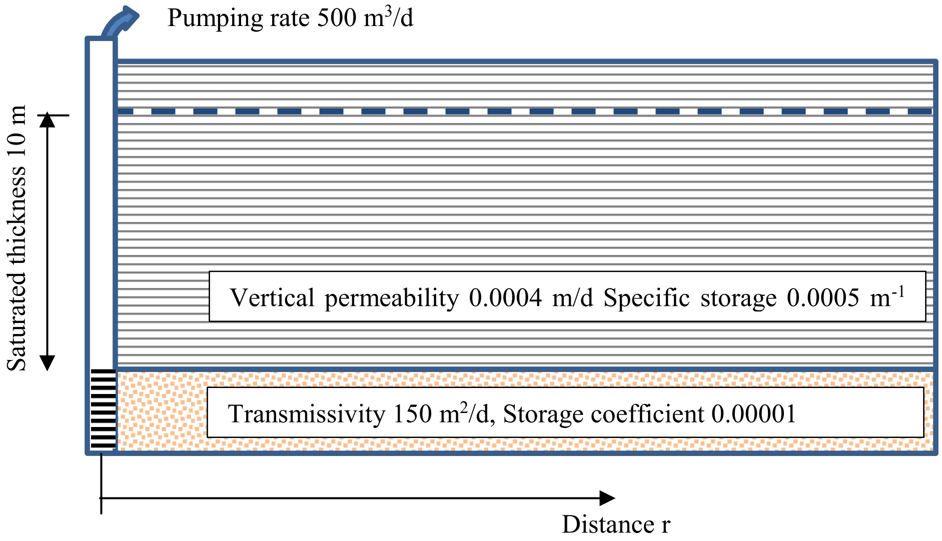

Figure 6: Example aquifer and consolidating layer case

Figure 7 shows the modelled drawdown in the base aquifer and the surface settlement for the case illustrated in Figure 6 for distances of 5 m and 20 m from the pumping bore.

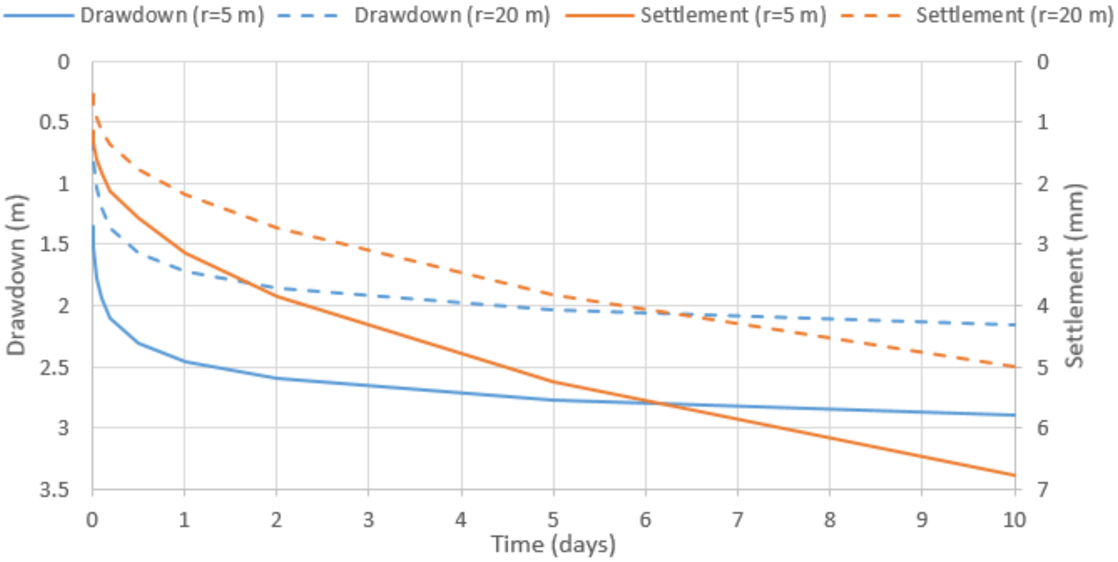

Figure 7: Modelled drawdown and settlement for sample case

The case illustrates that settlement continues to occur after drawdown as begun to stabilise. The settlement response with time provides a means of assessing the vertical consolidation parameters for the compressible formation as well as the transmissivity of the aquifer underlying the compressible unit. The utility of the approach depends upon the capacity to measure settlement to millimetre (or preferably sub-millimetre) resolution.

Assessment of parameters from test results can be carried out by testing trial choices against measured response.

4 INSTRUMENTATION

The measurements traditionally employed for interpretation of test pumping are:

- Measurement of the pumping rate over the duration of the test
- Measurement of piezometric level in the pumping bore and in piezometers near the pumping bore.
- Water quality can also be measured and can be helpful in assessing the origin of the water pumped and identifying the presence of contamination.

Measurements are made at regular intervals over the course of the test including a period after pumping has ceased. The results at the pumping bore during the recovery period after pumping ceases is particularly useful as groundwater recovery is not affected by the losses through the screen and formation in close proximity to the well as occurs during pumping.

Measurements of drawdown are typically made using pressure monitors installed in the pumping bores and the observation piezometers. These are typically recorded automatically by a downhole logger or by a telemetry system. These methods are well established.

Measurement of settlement is not typically carried out for pump tests but a number of methods are widely used for assessment of settlement for engineering projects and for large-scale response to groundwater extraction these include:

- Conventional survey using total station equipment – resolution to approximately 1 mm can be achieved over limited areas
- GPS based survey – resolution of several millimetres can be achieved over a wide area
- InSAR satellite based imaging – resolution at sub 10 mm resolution can be achieved over vast areas
- Tiltmeters – are employed during fracturing of coal seam gas bearing horizons to assess the extent of fracturing (seismic monitoring is also employed for this purpose but is not helpful for assessment of ground movement). Tiltmeters are commercially available with quoted resolution of 0.0013°.
- Hydrostatic profiler – a pressure measuring instrument is drawn through a tube containing water. The difference in pressure is a measure of relative elevation. This can resolve movement of about 2 cm.
- High resolution can be obtained using a partly fill pipe with high precision force transducers to record changes in level at selected location. These can achieve resolution of 0.025 mm. It requires a rigid pipe partly filled with water to a tightly controlled level corrected for temperature effects.
- Laser methods are available to measure the vertical movement of structures to about 1 mm resolution.

In addition to these methods, extensometers can be used to measure the settlement profile at particular locations.

In order to extend interpretation of test pumping results to include treatment of settlement methods for measurement of settlement to a resolution of 1 mm or better are required. In urban settings, creation of settlement in excess of about 20 mm for testing purposes would generally be unacceptable. Ideally monitoring settlement to sub-millimetre resolution at a multiple times per days would allow evaluation of response in stiff ground conditions.

Settlement of sub-millimetre resolution is not readily available to be routinely deployed for monitoring of test pumping but a range of methods offer promise. For the present conventional survey is considered to be the most practical method for monitoring of settlement for test pumping and thus the use of settlement monitoring to enhance test pumping analysis is limited to situations where settlement of up to 8 to 20 mm is anticipated. Where higher resolution is required use of high resolution tiltmeters offer the possibility of obtaining sub-millimetre resolution using a series of tiltmeters.

5 CONCLUSIONS

A method for enhancing interpretation of test pumping interpretation is provided. This offers potential to provide additional information regarding the consolidation properties of compressible units overlying aquifer, identification of directional response in groundwater systems and clearer indications of the location of hydraulic boundaries and extent of drawdown influence. The approach is considered to have immediate applicability where settlement of the order of 10 mm can be induced by pumping or where project needs can support use of high precision settlement monitoring. Use of tiltmeters offers the potential for interpretation of testing where settlement of only a few millimetres takes place. The method is illustrated for a practical case. The theory presented in this paper can readily be employed to assess settlement associated with deep excavations or pumping from deep aquifers.

6 ACKNOWLEDGEMENTS

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7 REFERENCES

Altamira. *Baseline report on InSAR monitoring on the Surat-Bowen Basin*, (2012) contained in Santos. *GLNG Gas Field Development Project's EIS Documents – Appendix AE-E Ground deformation monitoring and management program* – (https://www.statedevelopment.qld.gov.au/assessments-and-approvals/santos-glng-environmental-impact-statements.html, 22 August 2018))

Best, R.J. and Booker J.R. (2000). *Groundwater flow in layered systems*. Developments in Theoretical Soil Mechanics – The John Booker Memorial Symposium, Balkima.

Best R.J and Parker C.J. *Groundwater in Sydney – Tunnel inflows and settlement – Theory and experience*. 12th Australian Tunnelling Conference (2005). The Australian Underground Construction and Tunnelling Association. 2005.

Boulton, N.S. (1954). *Unsteady radial flow to a pumped well allowing for delayed yield from storage*. International Association of Science and Hydrology, Volume 2, pp 472-476, Rome.

Booker, J.R. and Best, R.J. (1990). *Analysis of layered aquifer systems*. The Institution of Engineers Australia, Conference on Hydraulics in Civil Engineering, pp 1-5. Sydney.

Gaver, D.P. (1996). (Jr) *Observing stochastic processes, and approximate transform inversion*. Operations Research, 14: 444-459.

Hoffmann, J., Galloway D.L., Zebker, H.A. (2003). *Inverse modeling of interbed storage parameters using land subsidence observations, Antelope Valley, California,* Water Resources Research 39 (2) 1031.

Kruseman, G.P. and De Ridder, N.A. *Analysis and evaluation of pump test data – Second Edition,* Publication 47, International Institute for Land Reclamation and Improvement, Wageningen, The Netherlands, 2000.

Neuman, S.P. (1975). *Analysis of pumping test data from anisotropic aquifers considering gravity response*. Water Resources Research. Volume 23, Number 8, pp 1683-1688.

Niu Wen-jie, Wang Zhenyu, Chen Feng and Li Hongran. (2013). *Settlement analysis of a confined sand aquifer overlain by a clay layer due to single well pumping*. Mathematical Problems in Engineering, Vol 2013. Hindawai Publishing Corporation.

Talbot (1939). A. *The accurate numerical inversion of Laplace transforms*. Journal of Institution of Mathematics Applications, Volume 23, pp 97-120.

Wen-jieNiu, Zhenyu Wang, Feng Chen, and Hongran Li, (2013). *Settlement Analysis of a Confined Sand Aquifer Overlain by a Clay Layer due to Single Well Pumping,* Mathematical Problems in Engineering Volume 2013.

8 APPENDIX – ANALYSIS OF MULTILAYER GROUNDWATER SYSTEMS

The background theory below follows the treatment presented by Booker and Best (1990) and Best and Booker (2000) similar approaches are described by Neuman (1975).

THEORY:

This paper provides an extension to the results presented in Booker & Best (1990) and includes treatment of a ring source and a circular source and takes account of multiple radial boundary conditions.

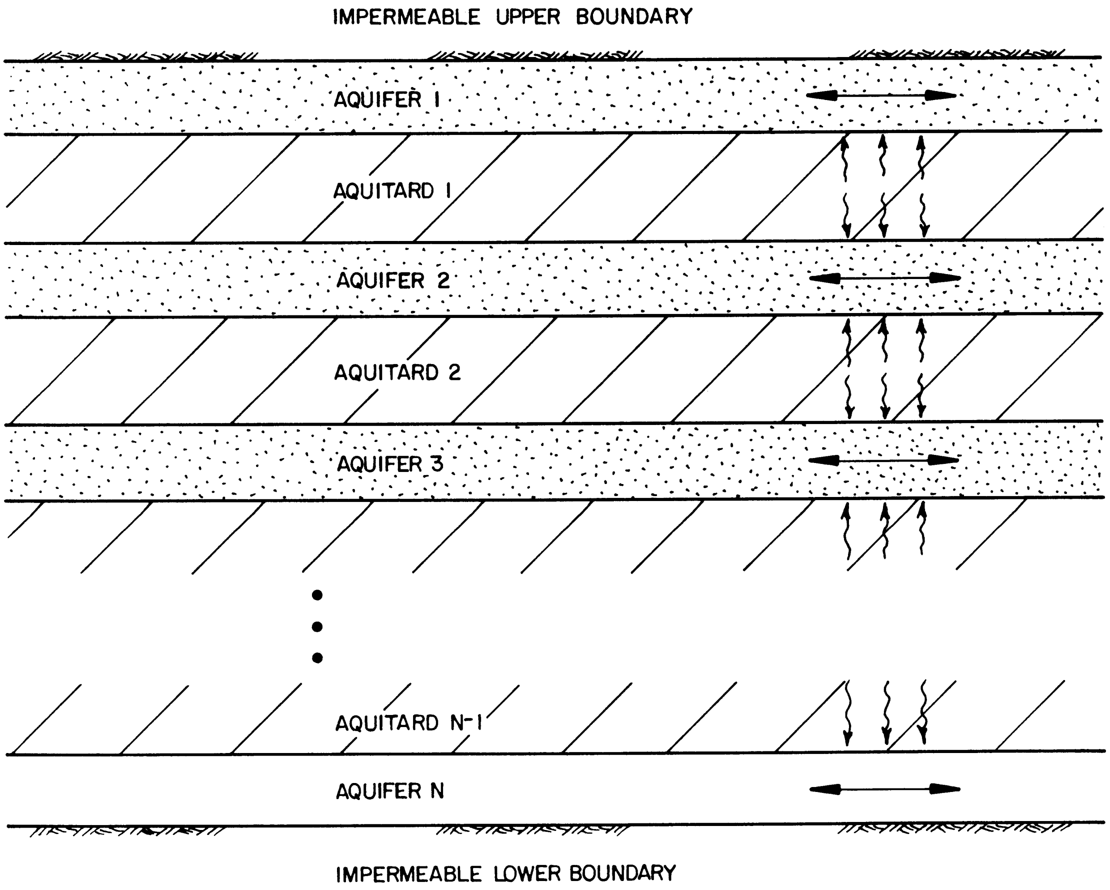

Figure 1A: Layered aquifer system.

8.1 FINITE LAYER APPROACH

The aquifer system is assumed to be made up on alternating aquifer and aquitard layers as shown in Figure 1. Flow is taken as purely horizontal in the aquifer layers and purely vertical in the aquitard layers. Impermeable boundaries are assumed above and below the layered system. Material properties are considered to be uniform within each layer. No account is taken of unsaturated flow.

The vertical flow within the aquitard unit below aquifer i is governed by Darcy's law:

$$k\frac{\partial^2\phi}{\partial z^2} = S_s\frac{\partial\phi}{\partial t} \tag{1}$$

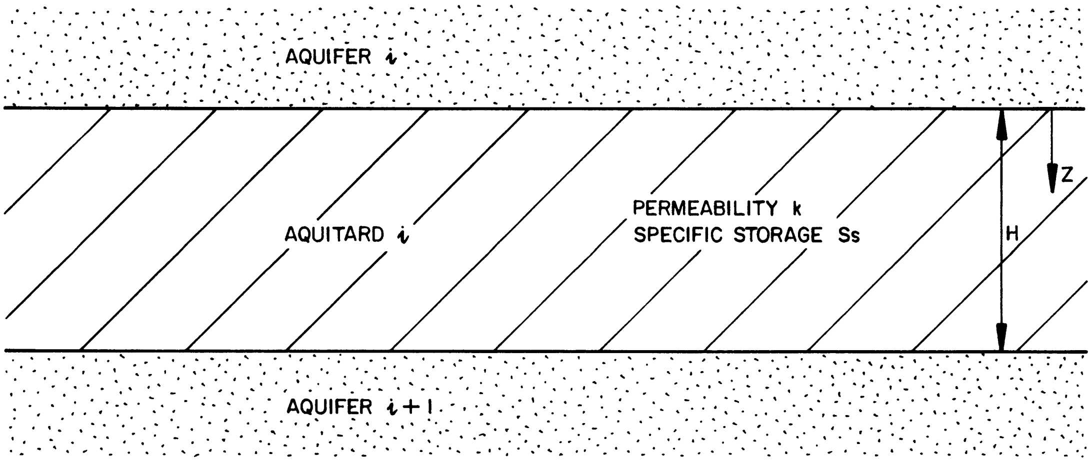

Figure 2A: Component aquitard

Where k , z and S_s are defined in Figure 2, t is time and ϕ is potentiometric head. Flow into the aquifer overlying the aquitard, q_i is given by:

$$q_i = k\frac{\partial \phi}{\partial z} \qquad \text{at} \quad z = 0 \tag{2}$$

Flow into the lower aquifer from the aquitard, q_{i+1}, is given by:

$$q_{i+1} = -k\frac{\partial \phi}{\partial z} \qquad \text{at} \quad z = H \tag{3}$$

Where H is the aquitard thickness. Taking the Laplace transform of Equation 1 and assuming that head is zero at time zero gives:

$$k\frac{\partial^2 \overline{\phi}}{\partial z^2} = S_s \rho \overline{\phi} \tag{4}$$

Where the superior bar indicates the Laplace transform and ρ is the Laplace transform variable. This has solution:

$$\overline{\phi} = \overline{\phi}_i \left(\cosh \mu z - \frac{\cosh \mu H \sinh \mu z}{\sinh \mu H} \right) + \overline{\phi}_{i+1} \left(\frac{\sinh \mu z}{\sinh \mu H}\right) \tag{5}$$

Where:

$\mu^2 = \frac{\rho S}{k}$ and μ is taken as having a positive real part

$\overline{\phi}_i$ is the Laplace transform of the head in the overlying aquifer

$\overline{\phi}_{i+1}$ is the Laplace transform of the head in the underlying aquifer

It follows that:

$$\begin{Bmatrix} \bar{q}_i \\ \bar{q}_{i+1} \end{Bmatrix} = -\frac{k\mu}{\sinh \mu H} \begin{bmatrix} \cosh \mu H & -1 \\ -1 & \cosh \mu H \end{bmatrix} \begin{Bmatrix} \bar{\phi}_i \\ \bar{\phi}_{i+1} \end{Bmatrix} \tag{6}$$

Assembling for each aquifer in the profile gives the following relationship between aquifer head and flow into the aquifers from the aquitards:

$$\begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \\ \vdots \\ \bar{q}_{n-1} \\ \bar{q}_n \end{Bmatrix} = -\begin{bmatrix} A_1 & B_1 & & & & \\ B_1 & A_1 + A_2 & B_2 & & & \\ & B_2 & A_2 + A_3 & & & \\ & & & \ddots & & \\ & & & & A_{n-2} + A_{n-1} & B_{n-1} \\ & & & & B_{n-1} & A_{n-1} \end{bmatrix} \begin{Bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\phi}_3 \\ \vdots \\ \bar{\phi}_{n-1} \\ \bar{\phi}_n \end{Bmatrix} \tag{7}$$

Using matrix notation Equation 7 becomes:

$$\{\bar{q}\} = -[A]\{\bar{\phi}\} \tag{8}$$

8.2 RADIAL FLOW

Assuming purely radial flow in the aquifers the equation for flow in aquifer i can be written:

$$T_i\left(\frac{\partial^2 \phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_i}{\partial r}\right) = -q_i + S_i\frac{\partial \phi_i}{\partial t} \tag{9}$$

Where

T_i is the transmissivity of aquifer i

S_i is the storage coefficient of aquifer i

r is the distance from the centre of axisymmetry of the radial flow

In matrix notation the system of equations relating head and aquifer inflow becomes:

$$[T]\left(\frac{\partial^2 \{\phi\}}{\partial r^2} + \frac{1}{r}\frac{\partial \{\phi\}}{\partial r}\right) = -\{q\} + [S]\frac{\partial \{\phi\}}{\partial t} \tag{10}$$

Where:

$[T]$ is the diagonal matrix of aquifer transmissivities

$[S]$ is the diagonal matrix of aquifer storage coefficients

$\{\phi\}$ is the vector of aquifer heads

$\{q\}$ is the vector of inflows into each aquifer

Taking the Laplace transform of Equation 10 and assuming zero head at time zero gives:

$$[T]\left(\frac{\partial^2\{\bar{\phi}\}}{\partial r^2}+\frac{1}{r}\frac{\partial\{\bar{\phi}\}}{\partial r}\right)=-\{\bar{q}\}+\rho[S]\{\bar{\phi}\} \tag{11}$$

Substituting Equation 8 gives:

$$[T]\left(\frac{\partial^2\{\bar{\phi}\}}{\partial r^2}+\frac{1}{r}\frac{\partial\{\bar{\phi}\}}{\partial r}\right)=[B]\{\bar{\phi}\} \tag{12}$$

Where $[B]$ is given by:

$$[B]=[A]+\rho[S] \tag{13}$$

Equation 12 has solutions of the form:

$$\{\bar{\phi}\}=X\ K_0(\omega r)\{v\}+Y\ I_0(\omega r)\{v\} \tag{14}$$

Where X and Y are constants and

K_0, I_0 are the zeroth order modified Bessel functions

ω^2 is an eigenvalue and

$\{v\}$ is the corresponding eigenvector satisfying

$$\omega^2[T]\{v\}-[B]\{v\}=0 \tag{15}$$

and ω is taken as having a positive real part.

The n eigenvalues and n eigenvectors of Equation 15 can be found analytically for $n<5$ and numerical methods can be employed for $n>4$. The solution to Equation 12 can be written:

$$\{\bar{\phi}\}=\sum_{i=1}^{n}\left(X_i\ K_0(\omega_i r)\{v_i\}+Y_i\ I_0(\omega_i r)\{v_i\}\right) \tag{16}$$

Where:

$\omega^2{}_i$ is the i'th eigenvalue of Equation 15

$\{v_i\}$ is the i'th eigenvector of Equation 15

X_i, Y_i are constants chosen to satisfy the radial boundary conditions

Rewriting Equation 16 in matrix form gives:

$$\{\bar{\phi}\}=[v][K_0(\omega r)]\{X\}+[v][I_0(\omega r)]\{Y\} \tag{17}$$

Where:

$[K_0(\omega r)]$ is a diagonal matrix with $K_0(\omega r)_{ii}=K_0(\omega_i r)$

$[I_0(\omega r)]$ is a diagonal matrix with $I_0(\omega r)_{ii}=I_0(\omega_i r)$

$[v]$ is the matrix of eigenvectors of Equation 15

8.2.1 Point source

For a series of point sources in each aquifer at $r=0$ of varying strengths with Laplace transforms given by $\{\bar{Q}\}$:

$$\{\overline{Q}\} = \lim_{r \to 0} -2\pi r[T]\frac{\partial\{\overline{\phi}\}}{\partial r}$$

$$\{\overline{Q}\} = \lim_{r \to 0} -2\pi r[T][v]\left[\frac{\partial K_0(\omega r)}{\partial r}\right]\{X\}$$

$$\{\overline{Q}\} = 2\pi[T][v]\{X\}$$

so that:

$$\{X\} = \frac{1}{2\pi}[v]^{-1}[T]^{-1}\{\overline{Q}\} \qquad (18)$$

Numerical inversion of the matrix of eigenvalues is not necessary due to the orthogonality of the eigenvectors with respect to $[T]$. Provided that the eigenvectors are scaled such that:

$$\{v_i\}^T\{v_i\} = 1 \qquad (19)$$

whence Equation 18 becomes:

$$\{X\} = \frac{1}{2\pi}[v]^T[T]^{-1}\{\overline{Q}\} \qquad (20)$$

The inverse of the diagonal matrix of transmissivities is simply the diagonal matrix of inverse transmissivities.

For an extensive aquifer system $\{\overline{\phi}\} = 0$ at great distance from the point source. This implies $\{Y\} = 0$. It follows that for point aquifer sources in an extensive aquifer system:

$$\{\overline{\phi}\} = \frac{1}{2\pi}[v][K_0(\omega r)][v]^T[T]^{-1}\{\overline{Q}\} \qquad (21)$$

For a source $\{Q_0\}$ that is constant with time the Laplace transformed source vector is given by:

$$\{\overline{Q}\} = \frac{\{Q_0\}}{\rho}$$

8.3 CALCULATION OF SETTLEMENT

Settlement can be calculated at the surface or at aquifer boundaries. Settlement at the surface is given by the sum of the settlements within each aquifer and aquitard. Aquifer settlement can be calculated by multiplying the head change by the aquifer storage coefficient. Aquitard settlement can be calculated as the integral of the flow to the overlying and underlying aquifers. Surface settlement is calculated as the sum of settlements from all the aquifers and aquitards:

$$\delta = \sum_{i=1}^{n} S_{si}\phi + \sum_{i=1}^{n}\int_0^t q_i \, dt \qquad (22)$$

In the Laplace transform domain the time integral of vertical flow into the aquifers simplifies to become the transformed vertical flow divided by the Laplace transform variable.

$$\overline{\delta} = \sum_{i=1}^{n} S_{si}\overline{\phi} + \frac{\overline{q}_i}{\rho} \qquad (23)$$

Settlements at particular times can be obtained by numerical inversion.