

ANALYTICAL MODELLING OF PAVEMENTS UNDER MOVING LOADS

Saina Emami¹ and Markus Oeser²

¹Graduate Geotechnical Engineer, Aurecon, Sydney, Australia ²Head of the Chair, Professor for Pavement Engineering, Aachen University, Germany

ABSTRACT

On a global scale, substantial financial resources are allocated for developing safe roads and the maintenance of existing ones. As a result, it is desirable to ensure there is no unnecessary spending and that allocated budgets are utilised as efficiently as possible. Conventionally, analytical pavement design and analysis takes place through the development of an axi-symmetric solution. These models are based on making several assumptions as well as transforming a given problem into an axi-symmetric system, which simplifies and undermines the accuracy of output that is achieved. CIRCLY is an example of a software package that evaluates pavement parameters through the utilisation of axi-symmetric solutions. In making these mathematical simplifications, models become restricted to axi-symmetric systems, stationary loads (i.e. a moving coordinate system is not considered), and static cases as the effects due to inertia are ignored. To eliminate these constraints, this paper aims to develop an alternative analytical approach which can be used to efficiently and accurately determine pavement responses under stationary and moving loads. The governing equation that forms the basis of the model is Cauchy's theorem for examining the deformation characteristics of a pavement structure. Several substitutions and simplifications have been made in order to incorporate a moving coordinate system and to transform Cauchy's theorem to a relationship that is entirely based on displacements. This will then allow for the calculation of associated stresses and strains. The relationship developed is solved through the utilisation of Fourier transformations and complex numbers, which is the main focus of this paper. Based on the derivations and associated formulations, a program has been developed which can be used to calculate solutions for flexible pavements under variable loading conditions. Several case studies have been carried out and the output analysed. It was concluded that, the method is able to address the assumptions made in conventional analytical models. Additionally, the program is capable of accurately carrying out calculations at substantially reduced computational effort, a main restriction that is encountered with numerical approaches.

1 INTRODUCTION

Pavements have an important role in the everyday lives of human beings and for this reason every year a substantial amount of government funds are allocated for the development of safer roads and the maintenance of existing ones. In the 2010-11 Australian budget, an all-time high of \$4.7 billion was granted for maintenance and upgrades to the NSW road network, which highlights their importance (BudgetNSW, 2010). On a global scale, it is no surprise that finances allocated for this purpose places substantial burdens on national budgets. Therefore, it is desirable to ensure that there is no unnecessary spending and that these finances are utilised as efficiently as possible.

The main design requirement of all pavement structures is the sustaining of a sufficient service life while maintaining high operational characteristics such as driver comfort. Therefore, pavements need to have sufficient bearing capacity, durability and surface characteristics, while providing driver safety in all conditions. To achieve the desired goals, considerable attention has been invested into improving material quality, construction practices and maintenance measures to improve all aspects of pavement design. Hence, it is essential to formulate approaches and tools that realistically represent the pavement structures being designed or analysed.

In 1977, L. J. Wardle developed a computer program known as CIRCLY, which is now widely used in Australia as the primary tool to analyse pavement layers. This program incorporates multi-layer elastic theory, in which the pavement is considered to be a structure with a number of horizontal layers, infinitely extending laterally with all layers, except for the subgrade, having a finite vertical thickness. Hence, the pavement is modeled as a multi-layer semi-infinite half space (Bodhinayake, 2008).

CIRCLY considers stationary circular loads, which can lead to an underestimation of stresses and strains of the pavement. In addition to this, pavement analysis is carried out by static analysis. A static analysis does not consider any inertia effects, and further disregards any deformation, which a system may experience after its initial loading. It should also be noted that not all materials, for instance asphalt, behave in an elastic manner, rather they are visco-elastic and in general, linear elastic multi-layer theory does not account for this difference.

As vehicles pass over a pavement, they exert loads that vary in space and time. Studies carried out by OECD (1992) have concluded that inherently, pavement loads are from the combined effect of three different forms of wheel loads

that are predominantly experienced by a pavement due to a vehicle. These include stationary (constant) wheel loads, moving constant wheel loads as well as moving dynamic wheel loads.

In practice, to be able to design and analyse the deformation characteristics of a structure, often dynamic experiments such as non-destructive impact tests are carried out to identify the material characteristics of existing pavement structures. These are then used for the determination of the associated pavement parameters, which is achieved through a back-calculation procedure. Many different algorithms and approaches have been, and continue to be developed to carry out this back-calculation. However, most of these approaches incorporate simplifications related to the material behaviour, and neglect to take into consideration the influence of factors such as the material inertia and load impact characteristics. Therefore, it can be concluded that results that are obtained from the utilisation of these models contain errors which need to be addressed.

By developing a computational model for a pavement structure, one is enabled to capture the structural properties of the various layers present within a system under analysis, allowing for the solving procedure to take place, in which the stress and deformation characteristics of the structure under analysis are examined and determined. In doing so, a solution to Cauchy's theorem for a layered structure is necessary for the deformation analysis, which can be obtained through the use of either numerical or analytical techniques, each which consists of assumptions or restrictions that limit their accuracy or applicability. Both methods will be discussed in this paper, with significant attention to the complex solution, being the primary focus. It should be noted that the information presented here is an extract of previous work carried out by the same authors under the same title.

2 COMPUTATIONAL MODEL

2.1 BACKGROUND

The computational model developed for the pavement aims to capture the structural properties of the various layers present within the system under analysis, allowing for the examination and determination of the stress and deformation characteristics of the structure. In doing so, an analytical approach has been taken in the formulation of the model, which ultimately consists of several Fourier integrals that are required to be evaluated.

2.2 CAUCHY'S THEOREM

Cauchy's Theorem can be adopted in examining the deformation characteristics of a structure. The instance where a flexible pavement of a certain number of layers exists and the load is a constant uniform pressure moving at a constant velocity v , can be modelled as follows,

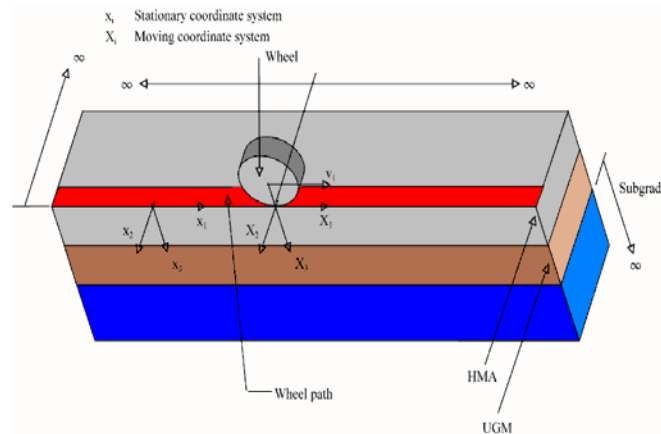


Figure 1: Pavement structure illustrating stationary and moving coordinate system

As can be seen in the diagram above, there are two different coordinate systems; the stationary coordinate system x_i and the moving coordinate system X_i , where $i=1,2,3$. In order to find the coordinates of the moving system with respect to the stationary system, one would have to make use of the relative displacement that exists between the two systems to establish a coordinate transformation as follows,

$$(x_1, x_2, x_3) \rightarrow (X_1 + vt, X_2, X_3)$$

Through adopting tensor notation, the stress matrix for a three dimensional body, which has a coordinate system of (x_1, x_2, x_3) is given by,

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad (1)$$

For the static case, where inertia effects are neglected due to small accelerations, the divergence of the stress is zero. In the dynamic situation, in which the mass of the material in combination with non-zero accelerations generates inertia, the effects due to inertia can add to the stress field. Here, the divergence of the stress must be equivalent to the inertia effects, and therefore,

$$\sigma_{ij,j} = \rho \cdot \ddot{u}_i \text{ where } \ddot{u}_i = \frac{d^2 u_i}{dt^2} \quad (2)$$

Where ρ is the density and \ddot{u}_i is the acceleration that is to be determined from the second derivative of the displacements, u_i with respect to time, t .

In observing the above relationship, the divergence of the stresses in the dynamic case is a differential operator with respect to time and space. Analysing such an equation is difficult and requires simplifications such as those regarding the system geometry, load, and material behaviour. Given a constant velocity, v , the problem at hand can be made time independent and therefore simplified from a physical point of view, reducing computational effort and time. The differential operators can be substituted as shown,

$$\sigma_{ij,j} = \rho \frac{d^2 u_i(x_1, x_2, x_3)}{dx_1 dt} \frac{dx_1}{dt} \quad (3)$$

For $t=0$, it is deduced that $x_1=X_1$ and therefore,

$$\sigma_{ij,j} = \rho \frac{d^2 u_i(X_1, X_2, X_3)}{dx_1^2} v_1^2 \text{ where } i = 1, 2, 3 \quad (4)$$

This is a differential equation that establishes a relationship between stresses and displacements. In order to simplify this, Cauchy's theorem can be equated against Hooke's law, which is only valid for linear elastic and linear visco-elastic problems.

$$\sigma_{ij,j} = (C_{ijkl} \epsilon_{kl})_{,j} \quad (5)$$

as follows,

$$\rho \frac{d^2 u_i(X_1, X_2, X_3)}{dx_1^2} v_1^2 = (C_{ijkl} \epsilon_{kl})_{,j} \quad (6)$$

Where C_{ijkl} is the material tensor, which must be a constant tensor that is independent of the stresses as well as the strains.

For small strains, the transformed Cauchy's equation will be a relation that is entirely based on the displacements,

$$\rho \frac{d^2 u_i(X_1, X_2, X_3)}{dx_1^2} v_1^2 = \left(C_{ijkl} \cdot \frac{1}{2} (u_{k,l} + u_{l,k}) \right)_{,j} \quad (7)$$

Once the displacements are obtained, they can be used to solve for the associated stresses and strains of the system.

2.3 MATHEMATICAL TREATMENT OF CAUCHY'S THEOREM

Conventional pavement analysis takes place by first determining displacements due to a load, and then carrying out an inverse analysis, determining the material characteristics. To achieve this, a solution to Cauchy's theorem for a layered structure is necessary, which can be obtained through the utilisation of either numerical or analytical techniques.

Numerically, methods such as the finite element method or the boundary element method can be utilised to solve for partial differential equations, including those within Cauchy's theorem. In comparison to the analytical method, the numerical method is applicable to many cases, geometrical restrictions do not apply and results are always obtained regardless of the boundary conditions. However, formulating a numerical model and carrying out analyses produce results that are of a lower quality as they are approximate and contain numerical errors. In addition to this, another drawback to the implementation of numerical analyses lies within the time consuming and computationally inefficient nature of the process.

Analytical models are capable of producing exact solutions in the instance where the boundaries of a pavement extend infinitely in the horizontal directions away from a load and the material behaves in a linear elastic manner. All other situations, such as the case where a load moves towards the edge of pavement or the pavement behaves any differently than linear elastic, the numerical techniques must be utilised. In the case of this paper, the pavement structure is assumed to comply with the situation in which the boundaries of the pavement extend infinitely away from the applied load and in which the material itself behaves in a linear elastic manner, and therefore, the formulation of an analytical

approach is considered to be an effective method to obtain solutions to the governing equations. It should be noted that the restrictions associated with the application of analytical models has been overcome through the use of a numerical approach, with appropriate consideration given to time and computational efficiency. However, since numerical techniques have not been adopted throughout the formulation process, results that are obtained are not considered to be approximations.

2.3.1 Axi-symmetric Solution

In order to carry out the analytical solution, two different approaches can be taken. In the first method, it is necessary to reduce the problem to an ordinary differential equation, which can then be solved through standard methods. This would require the elimination of two differential operators, which could be achieved by making several assumptions. The assumption that the geometry of the pavement and the loading is axi-symmetric, where the load has a circular shape and the boundaries extend infinitely from the centre of the load, means that at any distance away from this load, if a circle was to be drawn and all the points at the circumference were to be observed, no change in displacement would be occurring. Therefore, the derivatives of the displacement will be zero. By incorporating this assumption as well as transforming the differential equation into the polar coordinate system, the elimination of the $d\phi$ operator becomes possible and the differential equation will be based on the polar coordinates r and z .

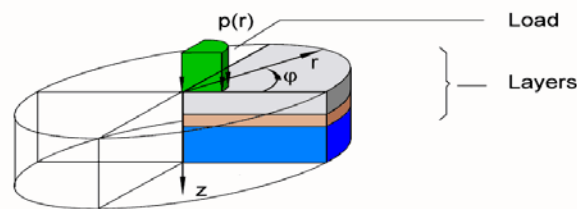


Figure 2: Polar coordinate system, axial symmetric loading and geometry.

Further reducing the problem by another differential operator can be achieved through the utilisation of Hankel's transformation method, in which only the forces that fulfill these two conditions are looked at, resulting in an ordinary partial differential equation that can be solved to determine the displacements and consequently the associated stresses and strains by standard methods.

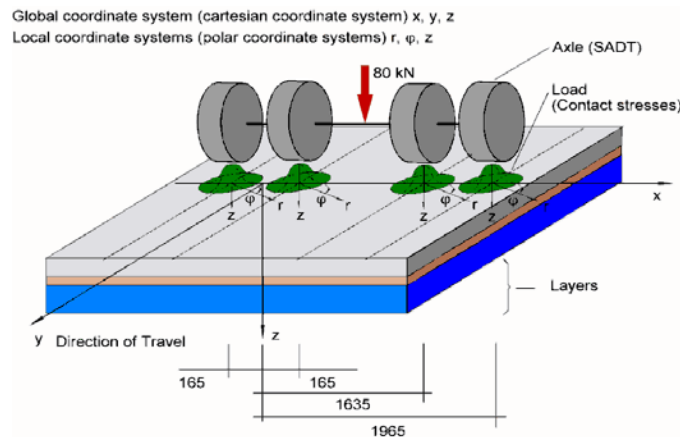


Figure 3: Analytical axi-symmetric concept for pavement analysis.

In making this mathematical simplification, the problem is now restricted to rotational or axi-symmetrical systems, to an elastic behaviour of the material, to stationary loads since a moving coordinate system has not been introduced and static cases as inertia is not being taken into consideration. Restrictions due to the linear elastic behaviour of the material are a consequence of using the superposition law, which are applied when looking at entire axle loads.

2.3.2 Complex Solution

The development of a different approach in solving the governing equations is desirable since in reality pavements can behave in a visco-elastic manner, and therefore loads are neither axi-symmetric nor stationary, and static cases are not plausible due to the effects of inertia. To achieve this, the Fourier transformation technique is utilised, which involves transforming the horizontal coordinates x_1 and x_2 from the real space into k_1 and k_2 of the Fourier space, and then solving the differential equation with the x_3 operator and incorporating the boundary conditions resulting from the layered structure. Boundary conditions in the horizontal directions x_1 and x_2 are not needed as it has been assumed that

they extend far enough to be considered as infinitely away from the load. However, since a pavement is a multi-layered structure, and the different layers must be considered, it is necessary to keep the x_3 factor in.

A tensor is a general expression to extend the various forms of mathematical numbers such as scalars, vectors and matrices, into higher orders. The Fourier transform A^* of any tensor A is defined by,

$$A(x_1, x_2, x_3) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A^*(k_1, k_2, x_3) e^{ik_1 x_1} e^{ik_2 x_2} dk_1 dk_2 \quad (8)$$

The first and second derivatives of Equation 8 are determined with respect to x_1 and x_2 . Additionally, since the partial differential equation includes differential expressions with respect to both dx_1 and dx_2 , it is required to also determine the derivatives of the two functions such that,

$$\frac{d^2 A(x_1, x_2, x_3)}{dx_1 dx_2} = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A^*(k_1, k_2, x_3) k_1 k_2 e^{ik_1 x_1} e^{ik_2 x_2} dk_1 dk_2 \quad (9)$$

In the Fourier space, the problem is that of a normal ordinary differential equation, in which the boundary conditions can be incorporated to then solve for the associated displacements, stresses and strains in the real space.

For linear elasticity, the material tensor for Hooke's law can be populated such that,

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (10)$$

Where λ and μ are the Lamé parameters and δ_{ij} is the Kronecker delta.

In homogenous, isotropic materials, the Lamé parameters satisfy Hooke's law in three dimensions. By incorporating λ and μ into Equation 10, the material tensor C_{ijkl} can be determined and the stress vector established as,

$$\underline{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} = \sigma_{21} \\ \sigma_{13} = \sigma_{31} \\ \sigma_{23} = \sigma_{32} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} + \varepsilon_{21} \\ \varepsilon_{13} + \varepsilon_{31} \\ \varepsilon_{23} + \varepsilon_{32} \end{Bmatrix} \quad (11)$$

Once the divergence of the stress tensor is determined, it is transformed into the Fourier space, the following is obtained,

$$\sigma_{ij,j}^* = -\frac{1}{4\pi^2} [A_{ij}] \begin{Bmatrix} u_1^* \\ u_2^* \\ u_3^* \end{Bmatrix} \frac{1}{\rho} + \frac{i}{4\pi^2} [B_{ij}] \begin{Bmatrix} \frac{du_1^*}{dx_3} \\ \frac{du_2^*}{dx_3} \\ \frac{du_3^*}{dx_3} \end{Bmatrix} \frac{1}{\rho} + \frac{1}{4\pi^2} [C_{ij}] \begin{Bmatrix} \frac{d^2 u_1^*}{dx_3^2} \\ \frac{d^2 u_2^*}{dx_3^2} \\ \frac{d^2 u_3^*}{dx_3^2} \end{Bmatrix} \frac{1}{\rho} \quad (12)$$

Where,

$$[A_{ij}] = \begin{bmatrix} -v_1^2 k_1^2 + (\lambda^* + 2\mu^*) k_1^2 + \mu^* k_2^2 & \mu^* k_1 k_2 + \lambda^2 k_1 k_2 & 0 \\ \mu^* k_1 k_2 + \lambda^2 k_1 k_2 & -v_1^2 k_1^2 + (\lambda^* + 2\mu^*) k_2^2 + \mu^* k_2^2 & 0 \\ 0 & 0 & -v_1^2 k_1^2 + k_1^2 \mu^* + k_2^2 \mu^* \end{bmatrix}$$

$$[B_{ij}] = \begin{bmatrix} 0 & 0 & \lambda^* k_1 + \mu^* k_1 \\ 0 & 0 & \lambda^* k_2 + \mu^* k_2 \\ \lambda^* k_1 + \mu^* k_1 & \lambda^* k_2 + \mu^* k_2 & 0 \end{bmatrix} \quad (13)$$

$$[C_{ij}] = \begin{bmatrix} \mu^* & 0 & 0 \\ 0 & \mu^* & 0 \\ 0 & 0 & \lambda^* + 2\mu^* \end{bmatrix}$$

By noting the velocity of the shear wave, C_s , and the velocity of the longitudinal wave, C_p , it is possible to simplify Equation 12. Since linear elasticity and linear visco-elasticity have been assumed, the coefficients of this system will be constant and not depend on such characteristics as the stresses or displacements.

To solve the problem explicitly for a single layered pavement structure, a Fourier transformation is carried out on this equation, which leads to a relation that is free from differentials, placing Cauchy's theorem completely in the Fourier space,

$$\sigma_{ij,j}^* = (-k_3^2 A_{ij} - k_3 B_{ij} - C_{ij}) u_j^* e^{ik_3 x_3} = 0_i \quad (14)$$

Noting that as long as there is a load there is a displacement and therefore the displacement component of Equation 14 cannot be zero, and further that an exponential can never be zero, it is possible to determine k_3 through solving that which is within the brackets against zero. Since this is a matrix, in order to set it to zero the determinant needs to be calculated and set to zero. Subsequently, this results in the following,

$$k_3 = \pm \sqrt{k_2^2 - k_1^2 \frac{v^2}{C_p^2} + k_1^2} \cdot i = \pm k_p i \quad (15)$$

$$k_3 = \pm \sqrt{k_2^2 - k_1^2 \frac{v^2}{C_s^2} + k_1^2} \cdot i = \pm k_s i \quad (16)$$

Introducing the Mach numbers into Equation 15 and Equation 16,

$$k_3 = \pm \sqrt{(1 - m_p^2) k_1^2 + k_2^2} \cdot i = \pm \kappa_p i \text{ where } m_p = \frac{v}{C_p} \quad (17)$$

$$k_3 = \pm \sqrt{(1 - m_s^2) k_1^2 + k_2^2} \cdot i = \pm \kappa_s i \text{ where } m_s = \frac{v}{C_s} \quad (18)$$

Hence, the eigenvalues for the compression and shear wave,

$$\kappa_p = \sqrt{(1 - m_p^2) k_1^2 + k_2^2} \quad (19)$$

$$\kappa_s = \sqrt{(1 - m_s^2) k_1^2 + k_2^2} \quad (20)$$

Once the eigenvalues have been solved for the compression and shear wave, the eigenvectors can be found, which is achieved by substituting the eigenvalues $\pm \kappa_p$ and $\pm \kappa_s$ in place of k_3 in Equation 14. The function that describes the displacement vector in the first transformation is assumed as one that can be expressed as a series of exponentials. The linear combination of these exponentials with the eigenvectors forms the complex displacement solution, u_i^* , where $i=1, 2, 3$. This can be represented in matrix form as follows,

$$\begin{pmatrix} u_1^* \\ u_2^* \\ u_3^* \end{pmatrix} = \begin{bmatrix} k_1 e^{-\kappa_p x_3} & 0 & \kappa_s e^{-\kappa_s x_3} & k_1 e^{\kappa_p x_3} & 0 & -\kappa_s e^{\kappa_s x_3} \\ k_2 e^{-\kappa_p x_3} & \kappa_s e^{-\kappa_s x_3} & 0 & k_2 e^{\kappa_p x_3} & -\kappa_s e^{\kappa_s x_3} & 0 \\ i \kappa_p e^{-\kappa_p x_3} & i k_2 e^{-\kappa_s x_3} & i k_1 e^{-\kappa_s x_3} - i \kappa_p e^{\kappa_p x_3} & i k_2 e^{\kappa_p x_3} & i k_2 e^{\kappa_s x_3} & i k_1 e^{\kappa_s x_3} \end{bmatrix} \begin{pmatrix} D_1^- \\ D_2^- \\ D_3^- \\ D_1^+ \\ D_2^+ \\ D_3^+ \end{pmatrix} \quad (21)$$

The above matrix can further be decomposed into two matrices separating the constants [C] from the exponentials [E].

Similarly, a matrix can be developed which relates to the stress characteristics of the pavement structure. For a pavement system, in an area where there is a stress, then the normal stress, σ_3^* , as well as the shear stresses, $\tau_{1,3}^*$ and $\tau_{2,3}^*$ must be in balance with the force from the outside,

$$\begin{pmatrix} \sigma_3^* \\ \tau_{1,3}^* \\ \tau_{2,3}^* \end{pmatrix} = \begin{bmatrix} -2\mu k_1 \kappa_p e^{-\kappa_p x_3} & -\mu k_1 k_2 & \kappa_s e^{-\kappa_s x_3} & k_1 e^{\kappa_p x_3} & 0 & -\kappa_s e^{\kappa_s x_3} \\ -2\mu k_2 \kappa_p e^{-\kappa_p x_3} & \kappa_s e^{-\kappa_s x_3} & 0 & k_2 e^{\kappa_p x_3} & -\kappa_s e^{\kappa_s x_3} & 0 \\ -\mu i (k_1^2 + k_2^2 + \kappa_s^2) e^{-\kappa_p x_3} & i k_2 e^{-\kappa_s x_3} & i k_1 e^{-\kappa_s x_3} - i \kappa_p e^{\kappa_p x_3} & i k_2 e^{\kappa_p x_3} & i k_2 e^{\kappa_s x_3} & i k_1 e^{\kappa_s x_3} \end{bmatrix} \begin{pmatrix} D_1^- \\ D_2^- \\ D_3^- \\ D_1^+ \\ D_2^+ \\ D_3^+ \end{pmatrix} \quad (22)$$

And therefore, in the general form, the complex displacement and stress values can be determined through,

$$\begin{Bmatrix} u \\ \sigma \end{Bmatrix} = [C][E](D) \quad (23)$$

In analysing a single layered pavement consisting only of a subgrade layer and extending infinitely from the surface, the above formulations can be directly applied to determine the complex displacement and stress solution. This is achieved through the incorporation of the boundary conditions, which depend on the external stresses of the pavement under analysis. At the surface, i.e. $x_3=0$, loads at specific velocities apply, resulting in the formation of associated stresses. Similarly, at the lower boundary of the pavement, i.e. $x_3=\infty$, it is known that there is no displacement, therefore allowing for the stresses to be determined. The implementation of these boundary conditions allows for the necessary computations to take place.

Extending the problem to one in which several layers of varying thicknesses exist necessitates variation of the above formulations such that it accounts for the interaction between the various layers, and consequently the impact these layers have on the stress and deformation characteristics of the pavement. Since a multi-layered pavement system is being considered, the boundaries will vary from layer to layer. For the uppermost layer, at the surface the boundary conditions are the loads and it is required to ensure that these loads are in balance with the applied stresses at the surface of this layer. When the layers below the uppermost one are considered, the condition to meet is for the contact stresses of the adjacent layers to not exhibit any displacement discontinuities. Finally, when considering the furthest layer from the surface, the subgrade, it is assumed that it extends infinitely from the applied load and since the effects due to the loads are negligible, the stresses and displacements at the base of the subgrade are taken to be of value zero.

Integral transformation techniques are referred to as analytical solutions when it is possible to solve them analytically and obtain a closed form solution to a problem. In the case of the one at hand, to simplify the problem in order to avoid the complicated mathematics, it is possible to utilise a numerical technique, which will only be used to solve the integral. Carrying out such techniques as the Gaussian integration method does not produce an approximate solution, but is considered as the simpler approach in solving the problem. Discrete values at specific points within the structure are obtained, as opposed to a closed form solution.

To integrate through this technique, the domain that is being integrated over is divided into sub-sections and the values of the function to be integrated at the Gauss points are determined. Once these are multiplied through with the Gauss weight factors and added together, an equivalent integral is obtained.

The difference between the proposed method and the previous studies that have been conducted lies within the fact that a numerical approach has not been adopted throughout the formulation process, and therefore results that are produced are not approximations. For instance, in analysing the problem of a multi-layered structure using the finite element method, to evaluate the partial differentials of Cauchy's theorem the system is split into smaller elements, the finite elements. Here, depending on the mesh, the displacements inside the element are determined by an interpolation function of the displacement values at the nodes. The shortcoming of this method lies in that the finite element assumes that for a certain domain, the displacements can be captured with a quadratic interpolation function, while the displacement field according to Cauchy's theorem is a polynomial of a much higher order. Observing that such assumptions have not been made in the proposed method, and that all the steps to solving the problem have been derived using exact solutions, it can be deduced that the technique employed throughout the formulation is not an approximation.

It should be noted that the constraint faced with the proposed method is due to the final step of the analysis in which a numerical integration technique is utilised and therefore only discrete values can be achieved. If the integral could be solved through an analytical technique, it would be possible to obtain a closed form solution that could be used to evaluate displacements anywhere within the system. However, due to the infinite boundary conditions and the complexity of the integral, it becomes too complicated to solve the problem analytically.

3 ANALYSIS

In order to carry out the formulations as developed in the computational model, a program has been developed by Prof. Dr. Markus Oeser (2011) through the use of Fortran and Visual Studio. This program incorporates the computational model outlined in conjunction with a constitutive model developed, which was based on the theory of rheology. It is capable of evaluating displacements, strains and stresses at discrete points within a pavement structure.

Many case studies were carried out on single-layered and multi-layered pavement systems under variable moving loads, structural and material specifications, and the results were compared against numerical examples that had been conducted on systems with the same characteristics.

As an example, a multi-layered pavement system with an Unbound Granular Material (UGM) acting as the sub-base was modelled with the characteristics summarised below

Table 1: Specifications for Multi-layered pavement system

Layer	Thickness	Vertical Modulus	Horizontal Modulus	Vertical Poisson	Horizontal Poisson
Asphalt	50	3000	3000	0.40	0.40
UGM	300	400	200	0.35	0.35
Subgrade	∞	80	40	0.45	0.45

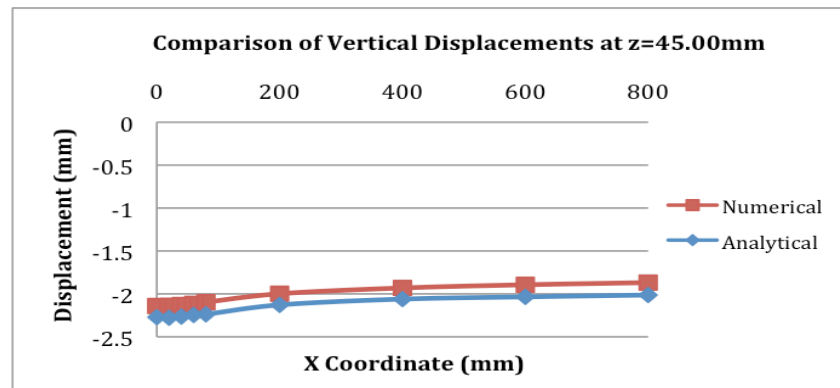
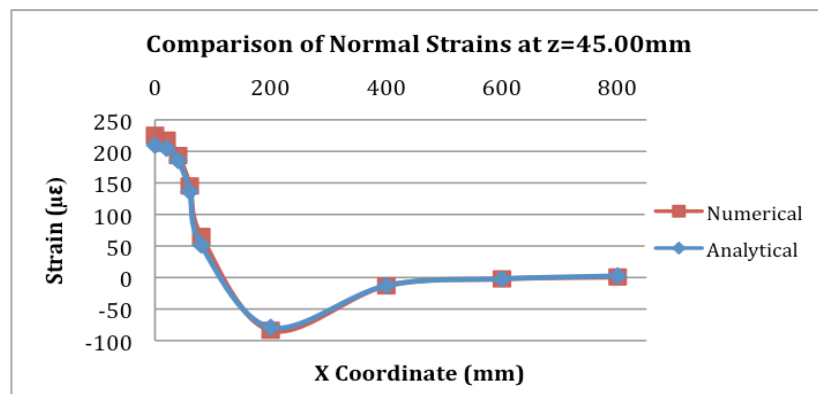
In addition to the above, the velocity of the moving load was specified at 50 km/hr, with a contact pressure of 750 kPa on a contact area of 92 mm diameter.

The specified coordinate for the output was set at a depth of 45.00 mm for the displacement, strains and stresses, which corresponds to bottom of the asphalt layer.

By carrying out this analysis, the displacement output which was obtained from the numerical and analytical method was again observed to follow the same path as each other but with a slight shift.

As can be observed, the output achieved through the analytical program closely follows on the same curve. However, it should be noted that a shift is expected in the two curves since the numerical analysis is carried out by meshing the structure and therefore has a finite boundary. This means that the effects due to the infinite thickness are not taken into consideration.

Further, the output achieved for the strains and stresses at the surface exhibited a very close superposition of each other.

Figure 4: Case 2 - Comparison of vertical displacements at $z=45.00$ mm.Figure 5: Comparison of normal strains and stresses experienced at $z=45.00$ mm.

The output obtained indicated a complete superposition for the strain and stresses evaluated at the specified depth.

4 CONCLUSIONS AND FUTURE RECOMMENDATIONS

The complex solution that has been developed addresses and eliminates the assumptions which have previously been employed in conventional analytical analysis for the design of pavement systems. In addition to this, it is capable of producing exact solutions at a substantially reduced computational time and effort when compared against numerical

analyses. Further, the program that has been developed from the mathematical model outlined is capable of evaluating pavement behaviour under static and moving loads, and analysis is carried out in a computationally efficient manner.

There are two drawbacks to the implementation of this analytical method. Firstly, the system becomes restricted to one in which the boundaries extend infinitely from the load. However, this does not pose as a major restriction as the loads will always in almost all situations be far enough away from the boundary so that the assumption of infinite boundaries does not become a problem. In the case where results at the boundary are required, it is possible to implement numerical methods as geometrical restrictions do not apply. Secondly, the material is assumed to behave linearly as loads are applied and removed. However, for the purpose of designing and analysing pavement systems this does not result in an issue as the design loads are within the linear range. Further developments in the proposed complex solution will be required in order to account for non-linear behaviour.

From the case studies carried out, it was found that by comparison, the output from both the numerical and analytical methods exhibit a very close superposition of each other with a fractional discrepancy error. In some instances, irregularities were observed in the plots, which can be attributed to the following factors,

- The Gaussian integration technique, which has been adopted in the back-transformation from the complex solution to the real space and has resulted in some singularities at the centre of loading, as well as inconsistencies with the output when compared with the numerical analysis
- Limited precision of computers in efficiently carrying out the complexity of the developed formulations

With regards to future recommendations, further research is suggested to be carried out in extension to the ideas presented in this paper. The successful implication of this technique can be applied to the design and analysis of other load bearing structures for transportation, such as for railway slab tracks. This is an area which has not been previously looked into in particular with regard to the impacts of moving loads. Therefore, further applications of similar studies could be implemented to be used in railway systems.

5 REFERENCES

- Bodhinayake, C. (2008), 'A study of non-linear behaviour of subgrades under cyclic loading for the development of a design chart for flexible pavements', PhD thesis, University of Wollongong, New South Wales, Australia
- OECD, IR2 (1992), 'Dynamic Loading of Pavements', Road Transport Research Programme, final report, OECD, Paris
- Oeser, M. and Pellinien, T. (2011), 'Computational framework for common visco-elastic models in engineering based on the theory of rheology', *Computers and Geotechnics*, vol. 42, pp. 145-156.

